Unit 2

Problem 1

1. Consider the class of polynomials $P(x) = ax^7 + bx^5 + cx^3 + dx$ for nonzero real numbers a, b, c, d.

   (a) What is the degree of $P(x)$? 7

Part-a

The degree is 7 because the highest exponent in x is 7

Part-b

(b) What is the leading term of $P(x)$? ax^7

The leading term is ax^7 because the degree is 7 (see previous explanation)

Part-c

(c) What is the leading coefficient of $P(x)$? a

The leading coefficient is a because the degree is 7 (see previous explanation)

Part-d

(d) How many zeros (counting real, imaginary and multiplicities) does $P(x)$ have? 7

There are 7 zeros because the degree of P(x) is 7.

Part-e

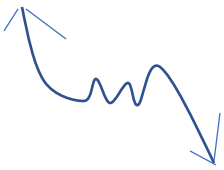
(e) What is the maximum number of turning points that $P(x)$ can have? 6

The degree is n = 7 so the maximum number of turning points is n-1 = 6

Part-f

(f) BONUS (2 points): Is it possible that $P(x)$ has no x-intercepts? (Yes/No) No

The end behavior of an odd-degree polynomial looks like

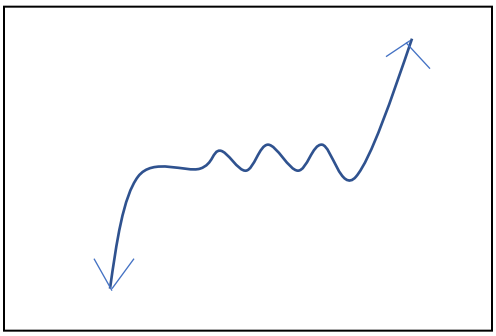
 or 

Based on the end behavior, any odd-degree polynomial function always crosses the x-axis.

Problem 2

2. Assuming $a > 0$, sketch the end behavior of the polynomial in Problem 1.
BONUS (2 points): Describe the end behavior symbolically.

Since the degree n = 7 is odd and the leading coefficient "a" is positive, then the graph rises to the right and falls to the left.



According to the sketch above,

As $x \to -\infty$, $f(x) \to -\infty$

As $x \to \infty$, $f(x) \to \infty$

Problem 3

3. Follow the instructions below to complete the problems on symmetry.

   (a) Determine if the polynomials in Problem 1 are even, odd or neither. Justify your answer.

Part-a

Given $P(x) = ax^7 + bx^5 + cx^3 + dx$,

$P(-x) = a(-x)^7 + b(-x)^5 + c(-x)^3 + d(-x)$

$= -ax^7 - bx^5 - cx^3 - dx$

$= -(ax^7 + bx^5 + cx^3 + dx)$

$= -P(x)$

Since $P(-x) = -P(x)$, then the polynomial in Problem 1 is odd

Part-b

(b) Use the symbolic test to prove $f(x) = \frac{|x|}{1+x^2}$ is even.

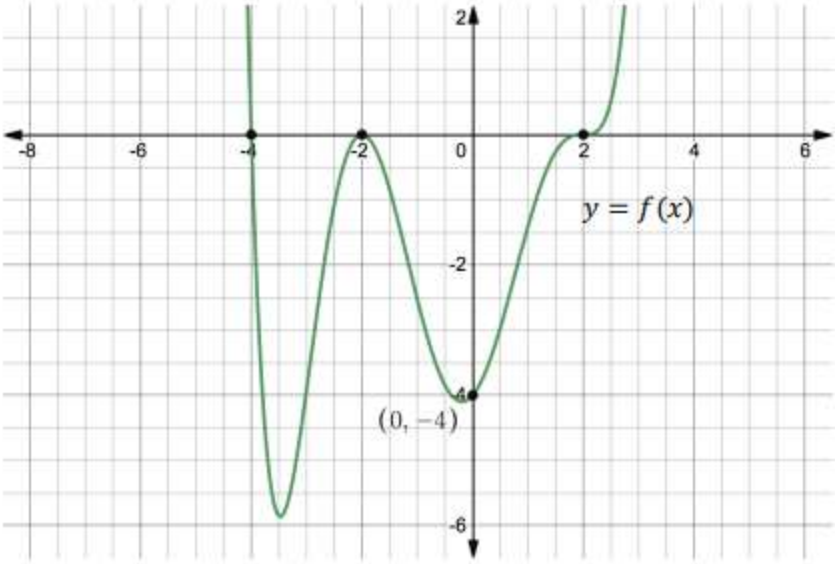
$f(-x) = \frac{|-x|}{1+(-x)^2}$

$= \frac{|x|}{1+x^2}$

$= f(x)$

Since $f(-x) = f(x)$, then the given function is even

Use the graph of the polynomial shown below to complete Problems 4-6.



Problem 4

4. Estimate the extrema of $f(x)$. Write your answers in the table below.

TYPE	Minimum	Maximum
Local	-5.9 and -4.1	0
Absolute	-5.9	None

The local minimums are -5.9 and -4.1 because we can find an open interval inside the domain of the function where the minimum possible value of the function are about -5.9 and -4.1.

More specifically, the minimum value of the function in the interval (-4, 2) is about -5.9 and the minimum possible value of the function in the interval (-2, 2) is about -4.1.

The local maximum is 0 because we can find an open interval inside the domain of the function where the maximum possible value of the function is 0. Specifically, in the interval (-3, 0), the maximum possible value of the function is 0.

The absolute minimum of the function is about -5.9 because the range of the function is from this value up to positive infinity.

The graph rises to the left and rises to the right so there is no absolute maximum.

Problem 5

5. Write the complete factored form of $f(x)$. (Assume the minimum degree possible.)

The zeros are -4, -2, and 2.

The corresponding factors are $(x+4)$, $(x+2)^2$, and $(x-2)$

We put an odd exponent (1) in the factor, when the graph crosses the x-axis at that zero.
On the other hand, we put an even exponent (2) in the factor, when the graph touches the x-axis at that zero.
There are n = 3 turning points so the minimum degree is n+1 = 4 and the complete factored form is
$f(x) = (x+4)(x+2)^2(x-2)$
The y-intercept is (0, -4) so we must solve for k in
$f(x) = k(x+4)(x+2)^2(x-2)$
$f(0) = k(4)(2)^2(-2) = -32k = -4$ so k = 1/8
Hence, $f(x) = \frac{1}{8}(x+4)(x+2)^2(x-2)$ assuming minimum degree possible

We can improve the estimate of the function by looking at how the graph crosses the x-axis at x = 2. It looks like we should use the exponent 3 in the factor (x-2) because the graph looks like a cubic function at x = 2.
$f(x) = k(x+4)(x+2)^2(x-2)^3$
We solve for k so that
$f(0) = k(4)(2)^2(-2)^3 = -128k = -4$ so k = 1/32 because the y-intercept is (0, -4)
Hence, $f(x) = \frac{1}{32}(x+4)(x+2)^2(x-2)^3$

Problem 6

6. Solve the inequality $f(x) < 0$ using the graph above. Write the solution set in interval notation.

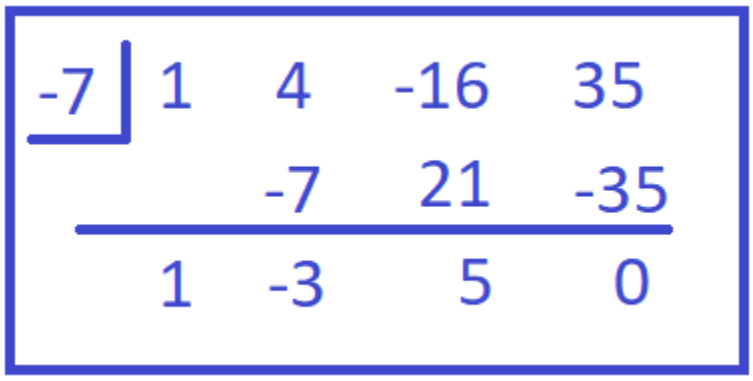
The graph is below the x-axis when x is between -4 and -2 and between -2 and 2.
Answer: $(-4,-2) \cup (-2,2)$

Problem 7

7. Use the rational zero test to find the zeros of the polynomial function: $f(x) = x^3 + 4x^2 - 16x + 35$. Use synthetic division to find the first zero and the quadratic formula to find the remaining two zeros.

The constant term 35 has factors 1, 5, 7, and 35 and the leading coefficient 1 has a factor 1.
Therefore, the possible rational zeros are $\pm 1, \pm 5, \pm 7, \pm 35$

Using synthetic division, we have



The quotient is $x^2 - 3x + 5$ with coefficients $a = 1$, $b = -3$, and $c = 5$

Using quadratic formula, we have

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(5)}}{2(1)}$$

$$= \frac{3 \pm \sqrt{-11}}{2} \text{ where } \sqrt{-11} = \sqrt{-1}\sqrt{-11} = \sqrt{11}\, i$$

$$= \frac{3}{2} \pm \frac{\sqrt{11}}{2} i$$

The zeros are -7, $\frac{3}{2} - \frac{\sqrt{11}}{2} i$, and $\frac{3}{2} + \frac{\sqrt{11}}{2} i$

Problem 8

8. For the given rational function, answer the following questions.

$$R(x) = \frac{2x + 10}{x^2 + 6x + 5}$$

(a) What is the domain of $R(x)$? $(-\infty, -5) \cup (-5, -1) \cup (-1, \infty)$

Part-a

Solve for x that will make the denominator zero

$x^2 + 6x + 5 = 0$

$(x + 5)(x + 1) = 0$

$x + 5 = 0$ or $x + 1 = 0$

$x = -5$ or $x = -1$

Hence, x cannot be equal to -5 or -1 and the domain is $(-\infty, -5) \cup (-5, -1) \cup (-1, \infty)$

Part-b

(b) What are the x-intercept(s) of $R(x)$, if any? None

Factor the numerator and denominator

$\frac{2(x+5)}{(x+5)(x+1)} = \frac{2}{x+1}$, $x \neq 5$

The numerator cannot be equal to zero so there is no x-intercept.

Part-c

(c) What is the y-intercept of $R(x)$?

(0, 2)

Evaluate $R(0)$

$R(0) = \frac{10}{5} = 2$ so the y-intercept is (0, 2)

Part-d

(d) Give the equation(s) for any vertical asymptotes of $R(x)$.

x = -1

The simplest form of the rational function is

$R(x) = \frac{2}{x+1}$, $x \neq 5$

The denominator of the simplifies function nis zero when x+1=0 or x = -1

Therefore, the vertical asymptote is x = -1

Part-e

(e) Give the equation(s) for any horizontal asymptotes of $R(x)$.

y= 0

The degree of the numerator is lower than the degree of the denominator (1 < 2), therefore, the horizontal asymptote is y = 0.

(f) Give the coordinates for any holes of $R(x)$.

(-5, -1/2)

The simplest form of the rational function is

$R(x) = \frac{2}{x+1}$, $x \neq 5$ because we cancelled the common factor (x+5).

Next, evaluate $R(5)$

$R(-5) = \frac{2}{-4} = \frac{-1}{2}$

Therefore, the hole is at (-5, -1/2)

Part-g

(g) BONUS (2 points): Prove that the graph of $R(x)$ does not intersect its horizontal asymptote(s).

The horizontal asymptote is y= 0 (See Part-e)

Solve for x so that $R(x) = 0$

$$\frac{2x + 10}{x^2 + 6x + 5} = 0$$

$$\frac{2(x + 5)}{(x + 5)(x + 1)} = 0$$

$$\frac{2}{x + 1} = 0$$

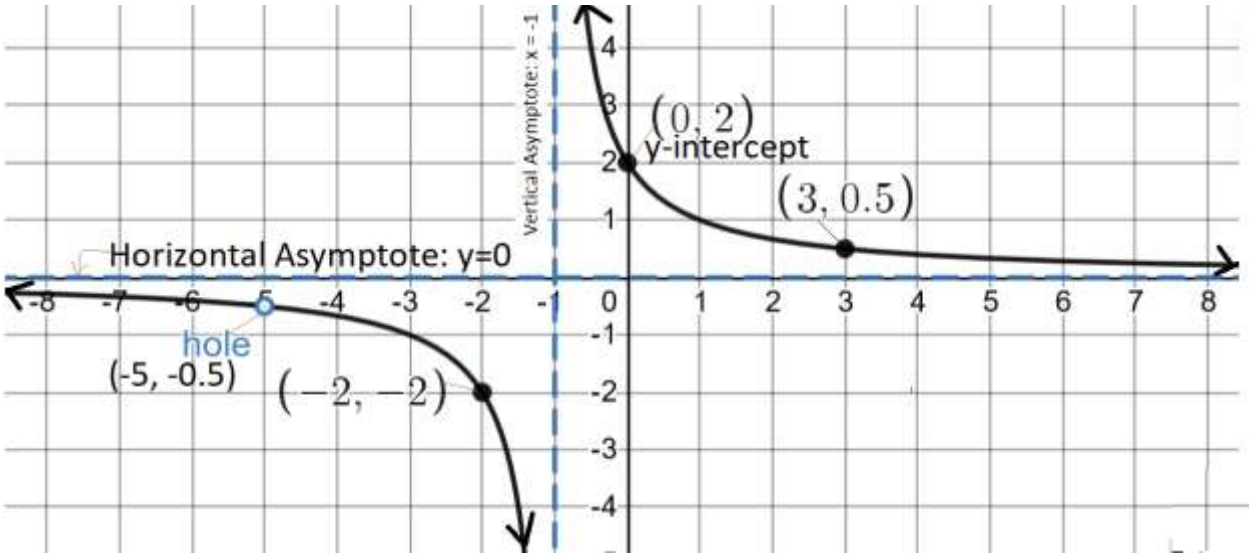
The numerator can never be equal to zero

Therefore, R(x) does not intersect the horizontal asymptote

Part-h

(h) Sketch the graph of $R(x)$ identifying all intercepts, asymptotes, holes and test points.

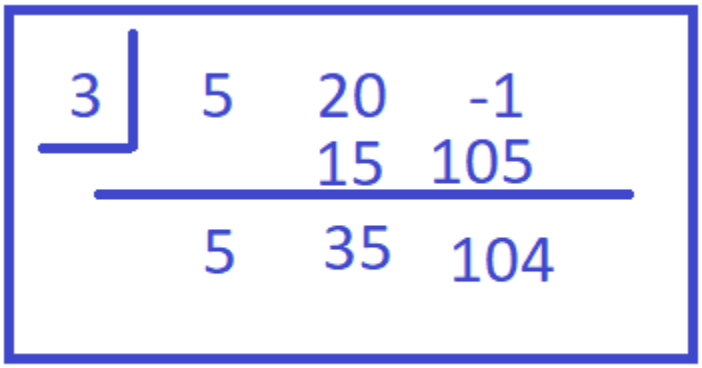
See the graph below



Problem 9

9. Find the slant asymptote of the rational function: $f(x) = \dfrac{5x^2 + 20x - 1}{x - 3}$

The degree of the numerator is one more than the degree of the denominator so the graph has a slant asymptote.
We use synthetic division to divide the numerator by the denominator.



According to the result of the synthetic division,

$f(x) = 5x + 35 + \dfrac{104}{x - 3}$

Therefore, the slant asymptote is $y = 5x + 35$

Problem 10

10. Consider the polynomial function: $P(x) = (x + 1)^2(x - 1)(x - 3)^2$.

(a) What are the zeros of $P(x)$?

Part-a

Equate each factor to zero and solve for x

$x + 1 = 0$ or $x - 1 = 0$ or $x - 3 = 0$

The zeros are -1, 1, and 3

Part-b

(b) State the multiplicity of the zeros of $P(x)$.

Based on the exponents of each factor, the multiplicities of the zeros -1, 1, and 3 are 2, 1, and 2, respectively.

Part-c

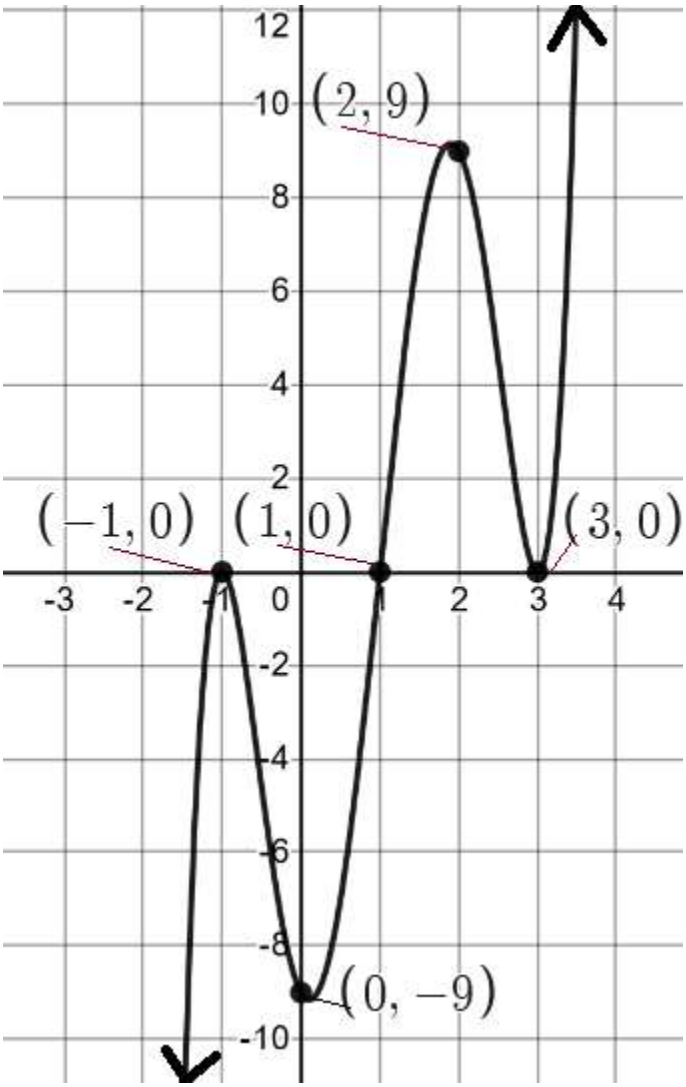
(c) Use the multiplicity test to determine whether $P(x)$ crosses or is tangent to the x-axis at its zeros.

P(x) is tangent to the x-axis at (-1, 0) and (3, 0) because the multiplicities of -1 and 3 are even numbers.

On the other hand, P(x) crosses the x-axis at (1, 0) because the multiplicity of 1 is an odd number.

Part-d

(d) Sketch the graph of $P(x)$ identifying all intercepts and test points.



Some Calculations:

$P(x) = (x+1)^2(x-1)(x-3)^2$

$P(2) = (2+1)^2(2-1)(2-3)^2$

$= (9)(1)(1)$

$= 9$

The graph contains the point (2, 9)

$P(0) = (0+1)^2(0-1)(0-3)^2$

$= (1)(-1)(9)$

$= -9$

The y-intercept is (0, -9)

The degree is 5 (odd number) and the leading coefficient is 1 (positive). Therefore, the end behavior is that the graph rises to the right and falls to the left.

Problem 11

11. The data in the table lists the number of monthly active Facebook users worldwide in billions.

Year	2015	2016	2017	2018	2019	2020
Users	1.59	1.86	2.13	2.32	2.50	2.80

(a) Use cubic regression to model the data. (Let $x = 0$ correspond to 2015, $x = 1$ correspond to 2016, etc.) Write down the equation below. Round to 4 decimal places.

Part-a

According to the calculator, the cubic regression equation that best fits the data is

$y = 1.5821 + 0.3546\,x - 0.0600\,x^2 + 0.0075\,x^3$

Part-b

(b) Use this model to predict users in 2021.

If x= 6, then

$y = 1.5821 + 0.3546\,(6) - 0.0600\,(6)^2 + 0.0075\,(6)^3$

≈ 3.17

The predicted number of users in 2021 is about 3.17 billions

Part-c

(c) Explain why cubic regression may or may not be better than linear or quadratic regression.

It seems that a linear regression is better for this data based on the scatter plot below. The points are very close to the fitted linear regression. We don't need the quadratic and cubic term in the model because the points do not follow a polynomial curve.

